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The Chebotarev theorem for ergodic toral automorphisms with respect to (G, ρ)–extensions

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Abstract. We consider distribution results for closed orbits of the partially hyperbolic system: an ergodic toral automorphism $\tilde{A}$ with respect to a (G, ρ)–extension A. In particular we obtain an analogue of the Chebotarev theorem in this situation which is an asymptotic formula for the number of closed orbits of the base transformation according to how they lift onto the extension space. To arrive at this result we introduce a cyclic extension $\hat{A}$ of A and deduce that $\hat{A}$ and A is essentially a group extension and homogeneous extension of $\tilde{A}$ respectively. This observation of a group extension is similar to the setting previously studied by Parry & Pollicott and using the prime orbit theorem of Waddington we then derive at an auxiliary result for the group extension analogoues to Parry & Pollicott. Finally we relate this auxiliary result to the homogeneous extension by resorting to the work of Noorani & Parry.

Keywords: ergodic toral automorphisms, (G, ρ)–extensions, lifting closed orbits, Chebotarev theorem.

Mathematical subject classification: 37C25.

1 Introduction

The classical distributional results of number theory, such as the prime number theorem and Chebotarev density theorem, has inspired a number of analogoues results in the context of counting closed geodesics and closed orbits in the field of dynamical systems and others (a good place to start is Parry & Pollicott [9] and references therein; Pollicott & Sharp [11], Anantharaman [1] for some recent additions). For the analogoues results in graph theory see Hashimoto [4]. In this note we are interested in closed orbits counting for the partially hyperbolic system, an ergodic (not necessarily hyperbolic) automorphisms of

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the finite-dimensional torus. Lind [5] call these system as quasihyperbolic automorphisms. In fact, we shall consider a (G, ρ)–extension A of an ergodic toral automorphism $\tilde{A}$ and provide a Chebotarev theorem for closed orbits of $\tilde{A}$ according to the way these orbits lift onto the extension space. The result presented here then extends the asymptotic formulae for (G, ρ)–extension of hyperbolic toral automorphisms previously obtained in Noorani & Parry [8]. In analogy with the classical Chebotarev theorem which uses the prime number theorem, the basic counting argument that is used here is based on the so-called prime orbit theorem which was derived earlier by Waddington [14]. Another approach is to use the analogue of Mertens' theorem (see [7], [13]).

The proof of our asymptotic formula is carried out in three steps. The first step is to classify how closed orbits from the base transformation lift onto the extension space. This is done via the notion of the degree of a closed orbit (see the work of Heilbronn in [2] for the motivation). The next step is to derive an auxiliary result by introducing a cyclic–extension $\hat{A}$ of A. It turns out that $\hat{A}$ is a group extension of $\tilde{A}$ (for some group action). In this case closed orbits of $\tilde{A}$ lift onto $\hat{A}$ according to their Frobenius classes (see Parry & Pollicott [10]). In particular the related Chebotarev Theorem for this extension applies. As in the hyperbolic case, it is in this stage that the analysis of the associated zeta and L-functions plays a crucial role in obtaining the required asymptotics. For toral automorphisms the required analytic properties can be derived directly by studying the eigenvalues of the associated matrix. This is where Waddington's result [14] comes in. (See Degli Esposti & Isola [3] for the situation of a hyperbolic toral automorphism.) The final step is to relate the notion of Frobenius classes and degree of the closed orbits which in turn is just a simple application of the result in [8].

In section 2, we classify the way closed orbits lift onto the extension space. We also introduce the aforementioned cyclic extension and consider appropriate identifications. The Chebotarev theorem is then derived for this extension in section 3. Finally we apply the result in §3, to obtain the Chebotarev theorem for our original dynamical systems.

2 (G, ρ)–Extensions and Identifications

We shall begin with the basic set-up of this paper (see also [8], [7]).

Let A be an ergodic automorphism of a d-dimensional torus T. For simplicity we shall also denote by A the $d \times d$ matrix generated by this automorphism. Note that A is an element of $GL(d, Z)$ with $\det A = \pm 1$ and A does not have any eigenvalue being a root of unity. Further information on the dynamical properties of this map can be found in the work of Lind [5].

Let m be fixed and G be the set of elements in T with 'order' m i.e. $G = \{g \in T : g^m = e_G\}$. It is clear that the restriction of A to G is a group isomorphism. Let us denote this restriction by ρ. Then by letting G act on the right of T and using multiplicative notation we have $A(xg) = A(x)\rho(g)$ for all $x \in T$, $g \in G$. In this case, we say G ρ–commute with A. Now, since G acts freely on T and $|G| < \infty$, we observe that A induces an automorphism $\tilde{A}$ on the quotient manifold $\tilde{T}$ $(= T/G)$, which is also an ergodic toral automorphism. Letting $\pi_G : T \to \tilde{T}$ be the covering map, we have $\pi_G A = \tilde{A}\pi_G$. We shall refer to A as a (G, ρ)–extension of $\tilde{A}$. Let $\tilde{\tau}$ be an $\tilde{A}$–closed orbit with (least) period $\lambda(\tilde{\tau})$ and τ be an A–closed orbit with period $\lambda(\tau)$ such that $\pi_G(\tau) = \tilde{\tau}$. It is not difficult to see that the period of τ is a multiple of the period of $\tilde{\tau}$. This motivates us to define the *degree* of τ over $\tilde{\tau}$ as the integer

$$deg\left(\frac{\tau}{\tilde{\tau}}\right) = \frac{\lambda(\tau)}{\lambda(\tilde{\tau})}.$$

Moreover if $\tau_1, \dots, \tau_t$ are the distinct A–closed orbits that covers $\tilde{\tau}$, then the following basic relation holds:

$$deg\left(\frac{\tau_1}{\tilde{\tau}}\right) + \ldots + deg\left(\frac{\tau_t}{\tilde{\tau}}\right) = |G|.$$

Now recall that a partition of a positive integer k is a collection of positive integers $l_1, l_2, \dots, l_t$ such that $k \geq l_1 \geq l_2 \geq \cdots l_t \geq 1$ and $l_1 + \cdots + l_t = k$. In this case we write $\underline{l}$ for the t–tuple $(l_1, \dots, l_t)$. Then the above basic equation gives us a partition of $|G|$. In this case we say $\tilde{\tau}$ *induces the partition* $\underline{l} = (l_1, l_2, \dots, l_t)$ of the integer $|G|$ if

$$\underline{l} = \left(deg\left(\frac{\tau_1}{\tilde{\tau}}\right), \dots, deg\left(\frac{\tau_t}{\tilde{\tau}}\right)\right) \qquad \text{(after reordering if need be).}$$

For each partition $\underline{l}$ of G, let $D_{\underline{l}} = \{\tilde{\tau} \subset \tilde{T} : \tilde{\tau}$ induces the partition $\underline{l}\}$. Moreover, let $\pi_{\underline{l}}(x) = \text{Card}\{\tilde{\tau} \subset \tilde{T} : \tilde{\tau} \in D_{\underline{l}},\ \lambda(\tilde{\tau}) \leq x\}$. Then the crux of this paper is to provide an asymptotic formula for $\pi_{\underline{l}}(x)$ as $x \to \infty$. To arrive at this result, we need to relate A and $\tilde{A}$ with a certain direct product dynamical system and do some identifications.

Since G is finite it is clear that $\rho^n = \text{id}$ for some positive integer $n = n(m)$. Now define the semi–direct product group $\hat{G}$ of Z_n and G as follows:

$$\hat{G} = Z_n \times_\rho G = \{(r, g) : r \in Z_n, g \in G\}$$

with operation

$$(r, g) \cdot (s, h) = (r + s, g \cdot \rho^r(h)).$$

It is easy to see that the identity element of $\hat{G}$ is $(0, e_G)$ and the inverse of (r, g) is $(-r, \rho^{-r}(g^{-1}))$.

We shall denote by $\hat{T}$ the direct product manifold $Z_n \times T$ and define $\hat{A}: \hat{T} \to \hat{T}$ as $\hat{A}(s, x) = (s+1, A(x))$. With this in mind, we shall refer to $\hat{A}$ as a Z_n–(cylic) extension of A. Introduce a right $\hat{G}$–action on $\hat{T}$ by $(s, x)(r, g) = (r+s, x\rho^s(g))$ for each $(r, g) \in \hat{G}$ and $(s, x) \in \hat{T}$. Then one can check that $\hat{G}$ acts freely on $\hat{T}$ and more importantly this action commutes with $\hat{A}$, i.e $\hat{A}\hat{g} = \hat{g}\hat{A} \quad \forall \hat{g} \in \hat{G}$. Form the $\hat{G}$–orbit space $\hat{T}/\hat{G}$ together with the induced automorphism. Then is straight–forward to check that this dynamical system is identifiable with $\tilde{A}: \tilde{T} \to \tilde{T}$. With this understanding, we shall refer to $\hat{A}$ as a (free) $\hat{G}$–extension of $\tilde{A}$. As mentioned in the introduction, this is the setting previously studied by Parry & Pollicott [10].

Now let $\hat{H}$ be the subgroup $Z_n \times \{e_G\}$ of $\hat{G}$ and consider the action of $\hat{H}$ on the space $\hat{T}$. The $\hat{H}$–orbit space is

$$\hat{T}/\hat{H} = \{(s, x) \cdot Z_n \times \{e_G\} : \; x \in T, r \in Z_n\}.$$

Since $(s, x)(r, e_G) = (r + s, x\rho^r(e_G)) = (r + s, x) \quad \forall (s, x) \in \hat{T}, (r, e_G) \in \hat{H}$ we have $\hat{T}/\hat{H} = \{(Z_n, x) \colon x \in T\}$. In particular $\hat{T}/\hat{H}$ is identifiable with T. Moreover, it can be checked that the induced map on $\hat{T}/\hat{H}$ is essentially A. Since $\hat{H}$ is not necessarily normal in $\hat{G}$, in the language of [8], A (with regard to the identification) is a $\hat{G}/\hat{H}$ (homogeneous)–extension of $\tilde{A}$.

It is helpful to have the following multi–commutative diagram in mind:

$$\begin{tikzcd}
\hat{T} \arrow[r, "\tilde{A}"] \arrow[dd, "\pi_{\hat{G}}"] & \hat{T} \arrow[dr] & & \\
 & & T \arrow[r, "A"] \arrow[dl] & T \\
\hat{T} \arrow[r, "\tilde{A}"] & \hat{T} & &
\end{tikzcd}$$

3 Lifting closed orbits of $\tilde{A}$ to $\hat{A}$

In this section, following Parry & Pollicott, we shall consider the $\hat{G}$–extension $\hat{A}$ of $\tilde{A}$ and derive an auxiallary result – the Chebotarev theorem – for this group extension.

Let $\pi_{\hat{G}}$ be the covering map of $\hat{A}$ onto $\tilde{A}$. Also let $\tilde{\tau}$ be an $\tilde{A}$–closed orbit with least period $\lambda(\tilde{\tau})$ and $\hat{\tau}$ an $\hat{A}$–closed orbit covering $\tilde{\tau}$, i.e $\pi_{\hat{G}}(\hat{\tau}) = \tilde{\tau}$. Then since the $\hat{G}$–action on $\hat{T}$ is free and commutes with $\hat{A}$ we deduce that there exists a unique element $\hat{\gamma} = \hat{\gamma}(\tilde{\tau}) \in \hat{G}$ such that if $p \in \hat{\tau}$, then

$$p\hat{\gamma} = \hat{A}^{\lambda(\tilde{\tau})}(p).$$

This group element $\hat{\gamma}$, which only depends on $\hat{\tau}$, is called a *Frobenius element* of $\hat{\tau}$. Moreover if $\hat{\tau}'$ is another $\hat{A}$–closed orbit also covering $\tilde{\tau}$ it is easy to see that the Frobenius elements of $\hat{\tau}'$ and of $\hat{\tau}$ are conjugate to each other. We shall call the conjugacy class determined by $\tilde{\tau}$, the *Frobenius class* of $\tilde{\tau}$ and is denoted by $[\tilde{\tau}]$. Note that if $\hat{\tau}_i$, $i = 1, \ldots, n$, are all the $\hat{A}$–closed orbit that covers $\tilde{\tau}$ then each $\hat{\tau}_i$ will share the same period $\ell\lambda(\tilde{\tau})$, where ℓ is the order of a Frobenius element of $\tilde{\tau}$. In fact $|\hat{G}| = n\ell$. Now given a point $xG \in \tilde{T}$, the following points $(0, x), (1, x), \ldots, (n-1, x) \in \hat{T}$ all lie above xG. The following result gives us the Frobenius element of these points if xG is a periodic point.

Proposition 1. *Let k be least such that $A^k(x) = xg$ for some $g \in G$. Then the Frobenius element of (r, x) is $(k, \rho^{-r}(g))$ for all $r = 0, 1, \ldots, n-1$.*

Proof. Since $\hat{G}$ acts freely on $\hat{T}$ and $\hat{G}$ commutes with $\hat{A}$, the Frobenius element (k, g') of (r, x) is given by $\hat{A}^k(r, x) = (r, x)(k, g')$. Thus $(r + k, A^k(x)) = (r + k, x\rho^r(g'))$. In particular $A^k(x) = x\rho^r(g')$. The fact that G acts freely on T then gives us $\rho^r(g') = g$. Taking inverse then gives the required result.

The zeta function of $\tilde{A}$ is defined by

$$\tilde{\zeta}(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \mathrm{Fix}_m \tilde{A}$$

where $\mathrm{Fix}_m \tilde{A} = \mathrm{Card}\{x \in \tilde{T} : \tilde{A}^m(x) = x\}$. Let h be the topological entropy of $\tilde{A}$. Then it is well–known that $\tilde{\zeta}(z)$ is analytic and non–zero for $|z| < e^{-h}$. We note that the zeta functions of A, $\zeta(z)$, and of $\hat{A}$, $\hat{\zeta}(z)$, are defined similarly. In particular since the topological entropy of $\tilde{A}$, A and $\hat{A}$ are the same (by virtue of finite extensions) we deduce that $\zeta(z)$ and $\hat{\zeta}(z)$ are analytic and non–zero for $|z| < e^{-h}$. The following result, which is due to Waddington [14], gives the necessary information regarding the analytic properties of $\tilde{\zeta}(z)$ for z beyond e^{-h}.

Proposition 2. *Let $\tilde{A}$ be an ergodic toral automorphism. Then*

$$\tilde{\zeta}(z) = \tilde{B}(z) \prod_{\alpha \in U} \frac{1}{(1 - e^h \alpha z)^{K(\alpha)}}$$

where $\tilde{B}(z)$ is analytic and non–zero for $|z| < \tilde{R}e^{-h}$ (some $\tilde{R} > 1$) and the elements of U are the terms in the expansion of $\prod_{\lambda}(1 - \lambda)$ (λ = eigenvalue of $\tilde{A}$

of modulus one) such that if $\alpha \in U$ *then* $K(\alpha)$ *is the coefficient of the term* α *in the above expansion.*

Moreover, using this result Waddington then obtained the prime orbit theorem for ergodic toral automorphisms.

Theorem 1 (Waddington [14]). *Let* $\tilde{A}$ *be an ergodic toral automorphism and let* $\pi(x) = |\{\tilde{\tau} : \lambda(\tilde{\tau}) \leq x\}|$. *Then*

$$\pi(x) \sim \frac{e^{h(x+1)}}{x} \sum_{\alpha \in U} K(\alpha) \frac{\alpha^{x+1}}{\alpha e^h - 1}$$

as $x \to \infty$.

Let χ be an irreducible representation of $\hat{G}$. The L–function of χ is defined as

$$L(z, \chi) = \exp \sum_{\tilde{\tau}} \sum_{n=1}^{\infty} \frac{z^{\lambda(\tilde{\tau})n} \chi([\tilde{\tau}]^n)}{n}$$

where the first sum is taken over all $\tilde{A}$–closed orbits. By comparing the above expression with $\tilde{\zeta}(z)$ we deduce that $L(z, \chi)$ is non–zero and analytic for $|z| < e^{-h}$. Moreover each $L(z, \chi)$ has a non-zero meromorphic extension to some disc of radius less than Re^{-h} some $R > 1$. Let $\mathrm{Irr}(\hat{G})$ be the collection of all irreducible characters of $\hat{G}$. Then as will be apparent later on, our main task is to obtain a detail analysis of the analytic properties of $L(z, \chi)$ for z slightly beyond e^{-h} for each $\chi \in \mathrm{Irr}(\hat{G})$. First observe that when $\chi = \chi_0$, the principal character, $L(z, \chi_0) = \tilde{\zeta}(z)$. Also from Serre [12]:

Proposition 3. *Let* $J = \{\eta_0, \eta_1, \ldots, \eta_{n-1}\}$ *be the irreducible characters of the cyclic group* Z_n. *Then there exists 1–dimensional characters* $\chi_i \in \mathrm{Irr}(\hat{G})$, $i = 0, \ldots, n-1$, *such that for each* $r \in Z_n$, $\chi_i(r, g) = \chi_i(r, e_G)$ $\forall g \in G$. *Moreover* $\chi_i(r, e_G) = \eta_i(r)$ $\forall r \in Z_n$.

The L–functions for these 'special' characters then satisfies:

Corollary 1. *Let* $K = \{\chi_0, \chi_1, \ldots, \chi_{n-1}\}$ *be as above. Then* $L(z, \chi_i) = \tilde{\zeta}(\omega^i z)$ *for all* $i = 0, 1, \ldots, n-1$, *where* ω *is a primitive* n*–th root of unity.*

Recall that the zeta function of $\hat{A}$ is denoted by $\hat{\zeta}(z)$.

Proposition 4. *Let* d_χ *be the degree of* $\chi \in \mathrm{Irr}(\hat{G})$. *Then*

$$\hat{\zeta}(z) = \prod_{\chi \in \mathrm{Irr}(\hat{G})} L(z, \chi)^{d_\chi}$$

Proof. We shall use the expansion

$$-\log(1-\mu^{\ell}) = \sum_{n=1}^{\infty} \frac{\mu^{n\ell}}{n},$$

where μ is a complex number. Let $g \in \hat{G}$ with order ℓ. Then by the orthoganality relation for characters and in particular $\sum_{\chi \text{ irreducible}} d_\chi^2 = |\hat{G}|$ we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\mu^{n\ell}}{n\ell} |\hat{G}| &= \sum_{n=1}^{\infty} \frac{\mu^{n\ell}}{n\ell} \sum_{\chi \text{ irreducible}} d_\chi^2 \\
&= \sum_{n=1}^{\infty} \frac{\mu^{n\ell}}{n\ell} \sum_{\chi} \overline{\chi}(e)\chi(g^{n\ell}) \\
&= \sum_{\chi} d_\chi \sum_{n=1}^{\infty} \frac{\mu^n \chi(g^n)}{n}
\end{aligned}$$

So $(1-\mu^{\ell})^{-|\hat{G}|/\ell} = \exp \sum_\chi d_\chi \sum_{n=1}^{\infty} \frac{\mu^n \chi(g^n)}{n}$. Take $\mu = z^{\lambda(\tilde{\tau})}$, $g = [\tilde{\tau}]$ and consider $\hat{\tau}_i$ where $\pi_{\hat{G}}(\hat{\tau}_i) = \tilde{\tau}$. Since $\lambda(\hat{\tau}_i) = \ell\lambda(\hat{\tau})$ we deduce that

$$(1-z^{\lambda(\hat{\tau}_i)})^{-|\hat{G}|/\ell} = \exp \sum_{\chi} d_\chi \sum_{n=1}^{\infty} \frac{z^{\lambda(\tilde{\tau})n} \chi([\tilde{\tau}]^n)}{n}$$

Recall that if $\pi_{\hat{G}}(\hat{\tau}_i) = \tilde{\tau}$ for $i = 1, \ldots, n$ then $n\ell = |\hat{G}|$. Therefore we obtain

$$\prod_{\pi_{\hat{G}}(\hat{\tau})=\tilde{\tau}} (1-z^{\lambda(\hat{\tau})})^{-1} \quad = \quad \exp \sum_{\chi} d_\chi \sum_{n=1}^{\infty} \frac{z^{\lambda(\tilde{\tau})n} \chi([\tilde{\tau}]^n)}{n}$$

Taking product over all $\tilde{A}$–closed orbits then completes this proof.

The following result whose proof is similar to the proof of Prop. 4.3 in Noorani & Parry [8] relates the zeta function of $\hat{A}$ and of A.

Proposition 5. *Let $\hat{\zeta}(z)$, $\zeta(z)$ be the zeta function of $\hat{A}$ and A respectively. Then*

$$\hat{\zeta}(z) = \prod_{i=0}^{n-1} \zeta(\omega^i z)$$

where ω is a primitive n-th root of unity.

Coupling the above two propositions we have the following result.

Proposition 6.

$$\prod_{i=0}^{n-1} \zeta(\omega^i z) = \prod_{\chi \in \mathrm{Irr}(\hat{G})} L(z,\chi)^{d_\chi}.$$

Applying Corollary 1 to Proposition 5, we have

Corollary 2.

$$\prod_{i=0}^{n-1} \zeta(\omega^i z) = \prod_{i=0}^{n-1} \tilde{\zeta}(\omega^i z) \prod_{\chi \in \mathrm{Irr}(\hat{G})/K} L(z,\chi)^{d_\chi}$$

Finally we have

Proposition 7. *There exists some $R > 1$ such that the function $L(z,\chi)$ is non–zero and analytic in the disc of radius $|z| < Re^{-h}$ for each χ not in K.*

Proof. By Cor. 2, Prop. 2 and using the fact that A, $\tilde{A}$ not only share the same topological entropy h but also the same eigenvalues including multiplicities (for A, $\tilde{A}$ are conjugate) we deduce that $\prod_{\chi \in \mathrm{Irr}(\hat{G})/K} L(z,\chi)^{d_\chi}$ is analytic and non–zero for $|z| < Re^{-h}$ some $R > 1$. The result then follows since $L(z,\chi)$ has a non-zero meromorphic extension to some disc of radius less than Re^{-h} some $R > 1$ for each χ.

The following is the main theorem of this section:

Theorem 2. *For a conjugacy class C of $\hat{G} = Z_n \times_\rho G$, let*

$$\pi_C(x) = \mathrm{Card}\{\tilde{\tau} \subset \tilde{T} : [\tilde{\tau}] = C, \lambda(\tilde{\tau}) \le x\}.$$

Then

$$\pi_C(x) \sim \frac{|C|}{|\hat{G}|} \frac{ne^{h(x+n)}}{x} \sum_{\alpha \in U} K(\alpha) \frac{\alpha^{x+1}}{\alpha e^{hn} - 1}$$

as $x \to \infty$ through the positive integers.

Proof. Let C be a conjugacy class of $\hat{G}$. To capture the $\tilde{A}$–closed orbits $\tilde{\tau}$ with Frobenius class $[\tilde{\tau}] = C$, we introduce the following zeta function:

$$\tilde{\zeta}_C(z) = \prod_{[\tilde{\tau}]=C} \left(1 - z^{\lambda(\tilde{\tau})}\right)^{-1}$$

where the product is taken over all $\tilde{A}$–closed orbits with Frobenius class C. Note that $\tilde{\zeta}_C(z)$ is just the restriction of $\tilde{\zeta}(z)$ to the $\tilde{A}$–closed orbits whose Frobenius class equals C. Let $g \in C$. By the orthogonality relation for irreducible characters we have

$$(|\hat{G}|/|C|)\log\tilde{\zeta}_C(z) = \sum_{\chi} \chi(g^{-1})\rho(z,\chi)$$

where $\rho(z,\chi) = \sum_{n=1}^{\infty}\sum_{\tilde{\tau}} \chi([\tilde{\tau}]^n) z^{\lambda(\tilde{\tau})n}/n$. Therefore by differentiating the above equation and applying Proposition 7 together with Corollary 1, we obtain

$$\frac{|\hat{G}|}{|C|}\frac{\tilde{\zeta}'_C(z)}{\tilde{\zeta}_C(z)} = \sum_{i=0}^{n-1} \chi_i(g^{-1})\frac{\tilde{\zeta}'(\omega^i z)}{\tilde{\zeta}(\omega^i z)} + \kappa(z)$$

where $\kappa(z)$ is some analytic function for $|z| < Re^{-h}$ some $R > 1$. Note that χ_i, $i = 0, 1, \ldots, n-1$ are essentially the characters of the cyclic group Z_n. The proof is completed by following similar calculations as in page 243-246 of Waddington [14] (which corresponds to the proof of Theorem 1 in this note) coupled with pages 141–142 of Parry & Pollicott [10].

Remarks.

1. When $U = \{1\}$ so that A and hence $\tilde{A}$ are hyperbolic, we recover theorem 4.5 of Noorani & Parry [8].

2. When $n = 1$ so that $A = \hat{A} = \tilde{A}$ we recover Waddington's prime orbit theorem, Theorem 1.

3. When $U = \{1\}$ and $n = 1$, we obtain the prime orbit theorem for hyperbolic toral automorphisms (see, for e.g., Prop. 2.3 of Degli Espositi & Isola [3]).

4 Applications to (G, ρ)–Extensions

In this final section, we come back to our original dynamical system which is the (G,ρ)–extension A of $\tilde{A}$. As mentioned in §2, we are interested in the behaviour of the counting function $\pi_{\underline{l}}(x) = \mathrm{Card}\{\tilde{\tau} \subset \tilde{T} : \tilde{\tau} \in D_{\underline{l}},\ \lambda(\tilde{\tau}) \le x\}$ where $D_{\underline{l}} = \{\tilde{\tau} \subset \tilde{T} : \tilde{\tau}$ induces the partition $\underline{l}\}$. We shall first characterise the set $D_{\underline{l}}$ for each partition $\underline{l}$ of $|G|$. To do this, we need the following notion:

Let K be another subgroup of $\hat{G}$. We can define a left action of $k \in K$ on the coset space $\hat{G}/\hat{H}$ by $k \cdot \hat{g}\hat{H} = k\hat{g}\hat{H}$. Let $K_1, \ldots, K_m$ be the distinct orbits of this action and r_i, $i = 1, \ldots, m$, be their respective 'sizes'. This different 'sizes' then form a partition of $|\hat{G}|/|\hat{H}|$. Note that G is identifiable with $\hat{G}/\hat{H}$. In this case we say K induces the partition $\underline{r} = (r_1, \ldots, r_m)$ of $|G|$ (after reordering if need be). It is easy to see that if k is conjugate to k' then the respective cyclic subgroups generated by them induces the same partition of $|G|$.

The proof of the following result can be found in [8] (Prop 2.1). We remark that the proof given there is for shift maps. The proof for toral automorphisms can be obtained via the obvious modifications.

Proposition 8. *Let $\tilde{\tau}$ be a $\tilde{A}$–closed orbit. Then $\tilde{\tau}$ induces the partition $\underline{l}$ of $|G|$ if and onlf if the action of the cylic group generated by some (and hence all) Frobenius element acssociated with $\tilde{\tau}$ induces the partition $\underline{l}$ on $|G|$.*

Let $C(g)$ denote the conjugacy class containing $g \in \hat{G}$. As an immediate corrolory to the above theorem, we have

Corollary 3. *Let $C_i = \{\tilde{\tau} \subset X : [\tilde{\tau}] = C(g_i)\}$ be the distinct subsets of $\tilde{A}$–closed orbits with Frobenius class $C(g_i)$ respectively such that the cylic subgroup generated by g_i induces the partition $\underline{l}$, $i = 1, \ldots, m$. Then*

$$D_{\underline{l}} = \bigcup_{i=1}^{m} C_i.$$

Hence by a direct application of Theorem 2 to the above corollary we have the main theorem of this paper which generalises Theorem 4.6 of [8].

Theorem 3. *Let A be a (G, ρ)–extension of $\tilde{A}$ and let $\underline{l}$ be a partition of $|G|$. Moreover let $D_{\underline{l}} = \{\tilde{\tau} \subset \tilde{T} : \tilde{\tau}$ induces the partition $\underline{l}\} = \bigcup_{i=1}^{m} C_i$ where $C_i = \{\tilde{\tau} \subset X : [\tilde{\tau}] = C(g_i)\}$, $i = 1, \ldots, m$. Then*

$$\pi_{\underline{l}}(x) \sim \frac{1}{|\hat{G}|} \sum_{i=1}^{m} |C(g_i)| \frac{n e^{h(x+n)}}{x} \sum_{\alpha \in U} K(\alpha) \frac{\alpha^{x+1}}{\alpha e^{hn} - 1}$$

as $x \to \infty$.

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